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LETTER TO THE EDITOR

Homogeneity and spectral dimension of aggregation fractals

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Abstract. The fracton or spectral dimension, d_s, of two Sutherland type 'ghost' aggregation fractals is obtained (for high space dimension) by a scaling argument. If, as seems probable, these fractals are homogeneous, the results ($d_s = 2 \ln 2/\ln 3$ and $d_s = 4/(\sqrt{17} - 1)$) support recent conjectures that d_s is close to $\frac{4}{3}$ for all homogeneous fractals. However, they are *not* consistent with the stronger conjecture that d_s is *exactly* $\frac{4}{3}$ for all such fractals.

Alexander and Orbach (1982) have observed that the fracton or spectral dimension, d_s, of the incipient infinite percolation cluster is independent of d, the space dimension, and approximately equal to $\frac{4}{3}$. This observation has been much discussed in the recent literature (Rammal and Toulouse 1983, Leyvraz and Stanley 1983, Meakin and Stanley 1983). It has been conjectured (with the support of numerical and analytical work on percolation clusters, lattice animals, and diffusion-limited aggregates) that $d_s = \frac{4}{3}$, either approximately or exactly, for all homogeneous fractals (Leyvraz and Stanley 1983, Pandey and Stauffer 1983, Havlin and Ben-Avraham 1983, Gould and Kohin 1984, Sahimi and Jerauld 1984, Wilke *et al* 1984, Meakin and Stanley 1983).

There is no really compelling reason to suppose that the relation is exact; however, the consequences would be profound if this were the case. So far as I know, no exactly calculable counterexample has been given. Any discussion remains tentative due to the absence (at present) of any obviously unique definition of homogeneity; one possible definition is proposed by Leyvraz and Stanley (1983). The qualitative feature sought is an absence of 'bottlenecks' (either systematic or random) blocking the path of a random walker on the cluster.

In this letter I argue that $d_s = \frac{4}{3}$ is probably *not* exact for all homogeneous fractals. This view is supported by consideration of the Sutherland 'ghost' or 'maximum chain' cluster (Sutherland 1967, 1970, Ball 1984, R C Ball and T A Witten, to be published). In high d, this type of fractal seems to be homogeneous; but simple arguments show it to have $d_s = 2 \ln 2/\ln 3 \simeq 1.26$. However, this value is sufficiently close to $\frac{4}{3}$ to support the weaker form of the conjecture, that $d_s \simeq \frac{4}{3}$.

The fractal arises in a hierarchical model of aggregation (the 'maximum chain' model of Sutherland (1970)). One starts with a set of point seeds (zeroth generation clusters). The nth generation of clusters is obtained by pairing the clusters of the previous generation; each pair is joined by a single bond or 'random weld'. All possible welds are taken with equal weight. The fractal is obtained in the limit $n \to \infty$. Clearly, this procedure generates only trees.

It has been shown (R C Ball and T A Witten, to be published) that whilst after each iteration the mass of a cluster has doubled, its radius of gyration has increased

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by a factor of $(3/2)^{1/2}$. Hence d_f, the fractal dimension (Mandelbrot 1982), is given exactly by

$$d_f = 2\ln 2/\ln(\tfrac{3}{2}) \simeq 3.4. \tag{1}$$

In addition, any pair of sites (R_i, R_j) on the cluster is connected by a unique sequence of N_{ij} bonds, which is a random walk in space. Hence

$$\langle (R_i - R_j)^2 \rangle \sim N_{ij}. \tag{2}$$

Now consider such a cluster embedded in a space of sufficient dimension that there are no double points (i.e. $d > 2d_f$; Mandelbrot 1982). This means that the electrical resistance, Ω_{ij}, between points i and j is given simply by

$$\Omega_{ij} = N_{ij}. \tag{3}$$

Combining (2) and (3), we see that the resistance $\Omega(R)$ between two points on the cluster separated by a distance R in space must scale as

$$\langle \Omega(R) \rangle \sim R^2.$$

It is possible to relate the resistance properties of the cluster to its spectral dimension by a scaling argument. Define $G(R, \omega)$, the average diffusion propagator (at frequency ω), between points on the fractal separated by R in space. This should have the scaling form

$$G(R, \omega) \sim R^x \exp[-R/\xi(\omega)]$$

where the diffusion length $\xi(\omega) \sim \omega^{-1/d_w}$. The RMS displacement $r(t)$ of a diffusant particle at time t is easily confirmed to obey $r^{d_w} \sim t$, in accordance with the usual definition of d_w. (Note that d_s is defined by $d_s = 2d_f/d_w$.)

By considering the generalised Laplace equation for a fractal object (Alexander and Orbach 1982), one can show that $G(R, 0) \propto \langle \Omega(R) \rangle$, and hence that $x = 2$. However, the conservation of total probability requires

$$\int G(R, \omega)\, e^{i\omega t}\, d^D R\, d\omega = \theta(t).$$

This imposes the (Einstein-type) scaling relation

$$d_w = d_f + x; \tag{4}$$

hence $d_s \equiv 2d_f/d_w = 2d_f/(d_f + 2)$.

Thus, for all $d > 2d_f \simeq 6.8$, one has by (1), $d_s = 2\ln 2/\ln 3 \simeq 1.262$. Since for these d values the cluster is a random tree without double points, it seems unlikely that it can be inhomogeneous in the sense of the definition proposed by Leyvraz and Stanley (1983)—i.e. containing exceptionally dense boundary sets that completely surround internal pieces of the fractal (thus hindering diffusion). Certainly it is hard to see how a tree can contain 'bottlenecks' in the usual intuitive sense. Thus it seems plausible, on physical grounds, that the cluster is indeed homogeneous. If so, its existence means that d_s does not equal $\frac{4}{3}$ exactly for all homogeneous fractals. The value of $d_s \simeq 1.26$ is, however, sufficiently close to $\frac{4}{3}$ to support the weaker conjecture of approximate equality. This is itself of interest, because the number of fractals for which accurate values of d_s are known is still quite small.

This number can be increased further by considering the 'polydisperse' Sutherland aggregate (Ball 1984). The scaling argument given above can be generalised to this case. The fractal is generated iteratively from a set of point seeds by taking one pair of clusters at random, welding them, and returning the resultant cluster to the set. The final structure has the resistance exponent $x = 2$ as before, but now $d_f = 4/(\sqrt{17} - 3)$, and hence, for $d > 2d_f$, $d_s = 4/(\sqrt{17} - 1) \simeq 1.28$.

The value of d_s for the 'monodisperse' ghost cluster can be interpreted by noting that the ensemble actually constitutes a subset of lattice animals of anomalously open structure (R C Ball and T A Witten, to be published). By selecting this subset, the value of d_f (which $= 4$ for animals at high d) is decreased whilst retaining the same resistance exponent, $x = 2$. Hence, by (4), the spectral dimension is changed from the animals ($d > 8$) value of $\frac{4}{3}$. This suggests that there may be some homogeneous fractals for which one can vary d_s continuously by altering d_f at constant x or *vice versa*. For a small enough change, one does not expect the homogeneity of the fractal to be lost. It remains to be seen whether an explicit example of this type can be found; since both fractals considered above have $d_s < \frac{4}{3}$, it would be interesting to see whether values of $d_s > \frac{4}{3}$ can be obtained in this way.

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